Math 2020 Tutorial 5

Outline

① Techniques for Evaluating Limit & Examples

② Directional derivative (Partial derivative)

③ A little bit about differential equations (depend on time)

Techniques for Evaluating Limit

· $\exists$ of limit & computing it.

1) By def ($\varepsilon - \delta$ argument, $\forall \varepsilon > 0, \exists \delta \cdots\cdots$)

2) By basic properties of limit

$\pm$, $\times$, $(\cdot)^n$ etc

(In general, continuous function preserve limits.

(i.e if $f(x)$ is continuous at $x = x_1$

then $\lim_{x \to x_0} f(x) = f(\lim_{x \to x_0} x) = f(x_0)$)

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} (x+1)$$

3) By Squeeze Thm

4) By L'Hopital Rule

5) By taking $\lim_{r \to 0^+}$ in polar coord. (r, θ)

for calculating $\lim_{(x,y) \to (0,0)}$ on $\mathbb{R}^2$.

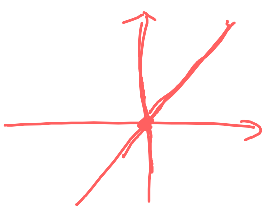
$x = r\cos\theta$

$y = r\sin\theta$.

· $\nexists$ of limit

1) By def ($\exists \varepsilon > 0, \forall \delta > 0, \cdots\cdots$)

2) By finding different paths to get unequal limits.



$y = x$

$y = 1$

$x = 0$

Case study:

Limit of a rational funct.

$$f(\vec{x}) = \frac{P(\vec{x})}{Q(\vec{x})} \quad \text{at} \quad \vec{a} \in \mathbb{R}^n$$

here P. Q polynomials on $\mathbb{R}^n$, $Q(\vec{x}) \not\equiv 0$.

Case ①: $Q(\vec{a}) \neq 0$

$\because$ $f(\vec{a})$ is well-defined & rational function are continuous,

$\therefore$ $\lim_{x \to \vec{a}} f(\vec{x}) = f(\vec{a}) = \frac{P(\vec{a})}{Q(\vec{a})}$

② case $Q(\vec{a}) = 0$ (for $n=1$ $\lim_{x\to 0} \frac{P(x)}{x}$ w/ $P(x)$ is polynomial.

subcase i) $P(\vec{a}) \neq 0$ ✱ (take $P(x) = 1$)

Let define $g(\vec{x}) = \frac{1}{f(\vec{x})} = \frac{Q(\vec{x})}{P(\vec{x})}$

Apply case ① on $g(\vec{x})$, $\lim_{x\to a} g(\vec{x}) = g(\vec{a}) = \frac{Q(\vec{a})}{P(\vec{a})} = \frac{0}{P(\vec{a})} = 0$

we have $\lim_{x\to\vec{a}} f(\vec{x})$ does not exist in $\mathbb{R}$.

subcase ii) ✱ $P(\vec{a}) = 0$. Any outcome is possible. depend on behavior of zeros of $P(\vec{x})$ & $Q(\vec{x})$

Subcase I $\quad n=1$

e.g. Evaluate $\lim\limits_{x\to 1} \dfrac{x^2-4x+3}{x^2-1}$

$P(x)$

$Q(x)$

$\lim\limits_{x\to 1} Q(x) = 0 = \lim\limits_{x\to 1} P(x)$

method ① $\lim\limits_{x\to 1} \dfrac{x^2-4x+3}{x^2-1} \overset{\text{L'Hopital}}{=} \lim\limits_{x\to 1} \dfrac{2x-4}{2x} = \dfrac{2\cdot 1-4}{2\cdot 1} = -1$

method ② $\lim\limits_{x\to 1} \dfrac{x^2-4x+3}{x^2-1} = \lim\limits_{x\to 1} \dfrac{(x-1)(x-3)}{(x+1)(x-1)} = \lim\limits_{x\to 1} \dfrac{x-3}{x+1} = \dfrac{1-3}{1+1} = -1$

$\tilde{P}(x)$

$\tilde{Q}(x)$

Subsubcase Ⅱ $\quad n=2$

e.g. Evaluate $\displaystyle \lim_{(x,y)\to(1,0)} \frac{4xy^2-x^2-5y^2+2x-1}{-x^2y+xy}$ $P(\vec{x})$ $Q(\vec{x})$

$$\lim_{(x,y)\to(1,0)} P(\vec{x}) = 0 = \lim_{(x,y)\to(1,0)} Q(\vec{x})$$

by Translation

$$\lim_{(x,y)\to(1,0)} \frac{4xy^2-x^2-5y^2+2x-1}{-x^2y+xy}$$

Substitute x by $x+1$ $\quad x\to 1 \quad x+1 \quad \lim_{x\to 0} x+1=1$

$$= \lim_{(x,y)\to(0,0)} = \frac{4(x+1)y^2-(x+1)^2-5y^2+2(x+1)-1}{-(x+1)^2y+(x+1)y}$$

$$= \lim_{(x,y)\to(0,0)} = \frac{-4xy^2+x^2+y^2}{x^2y+xy}$$

$(x,y)\to(0,0) \quad (n=2) \iff r\to 0$

polar coordinate.

$$= \lim_{r\to 0} \frac{-4r^3\sin^2\theta\cos\theta+r^2}{r^3\sin\theta\cos^2\theta+r^2\sin\theta\cos\theta} = \lim_{r\to 0} \frac{-4r\sin^2\theta\cos\theta+1}{r\sin\theta\cos^2\theta+\sin\theta\cos\theta}$$

Does not exist.

$r\to 0 \quad \theta=\frac{\pi}{4}. \qquad r\to 0 \quad \theta=\frac{\pi}{6}$

☆ Observation

- After transforming problem into finding limit

$$\lim_{(x,y)\to(0,0)} \frac{\overbrace{-4xy^2}^{3} + \overbrace{\boxed{x^2+y^2}}^{2} = P(x)}{\underbrace{x^2y}_{3} + \underbrace{\boxed{xy}}_{2} = Q(\vec{x})} \text{ "=" } \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{xy}$$

Only terms of the lowest deg in the polynomials

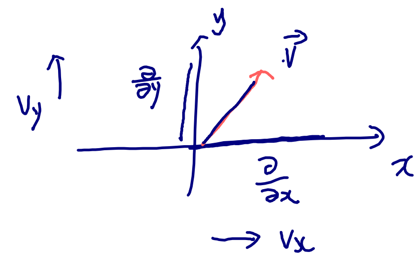
(i.e x^2+y^2 , xy in our e.g)

would determine the result

Directional Derivative. (Partial derivative)

Motivation: $n=1$ (Partial derivative = derivative)

$n=1$



Def: Let $\mathbb{R}^n \supset \underset{\text{open}}{\Omega} \xrightarrow{f} \mathbb{R}$, $\vec{x} \in \Omega$ & $\vec{v} \in \mathbb{R}^n$

The directional derivative of f along $\vec{v}$ at $\vec{x}$,

denoted by $\partial_{\vec{v}} f(\vec{x})$ or $\frac{\partial f}{\partial \vec{v}}(\vec{x})$, is defined as

$$\frac{\partial f}{\partial \vec{v}}(\vec{x}) := \lim_{t \to 0} \frac{f(x + t\vec{v}) - f(\vec{x})}{t}$$

In fact $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ partial direvative is a special case of

directional derivative.